\section{Composite systems}

It is well known that if the states of a pair of systems $S_1$ and $S_2$ are represented by points in Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then the state of the composite system $S = S_1 \times S_2$ (the two systems $S_1$ and $S_2$ regarded as a single system) are represented correctly by points of the product $\mathcal{H}_1 \times \mathcal{H}_2$. Thus if $\{i\}$ is a complete orthonormal set for $\mathcal{H}_1$ and $\{j\}$ for $\mathcal{H}_2$, the general state of $S = S_1 \times S_2$ has the form:

\begin{equation}
\psi_S = \sum_{ij} a_{ij} \phi_i \otimes \phi_j \quad \text{where} \quad \sum_{ij} a_{ij}^* a_{ij} = 1
\end{equation}

In this case we shall call $a_{ij}^* a_{ij}$ the joint probability amplitude distribution of $\psi_S$ over $\mathcal{H}_1$ and $\mathcal{H}_2$. On the standard probabilistic interpretation $a_{ij}^* a_{ij}$ represents the joint probability that $i$ will be found in state $\phi_i$ and $j$ will be found in $\phi_j$. Following the probabilistic model we now derive some distributions from a state $\psi_S$. Let $A$ be an operator in $S$ with eigenfunctions $\phi_i$ and eigenvalues $\lambda_i$. Conjugation in $S$, with eigenfunctions $\phi_i$, eigenvalues $\lambda_i$. Then the joint distribution of $\psi_S$ over $\mathcal{H}_1$ and $\mathcal{H}_2$ is:

\begin{equation}
\rho_{ij} = \rho(\phi_i, \phi_j) = |(\phi_i \otimes \phi_j, \psi_S)|^2
\end{equation}

The marginal distributions of $\psi_S$ over $\mathcal{H}_1$ and of $\psi_S$ over $\mathcal{H}_2$ are:

\begin{align}
\rho_i &= \rho(\phi_i) = \sum_j |(\phi_i, \psi_S)|^2 \\
\rho_j &= \rho(\phi_j) = \sum_i |(\phi_i, \psi_S)|^2
\end{align}
Notation $[\phi_i] = \text{prob on } \phi_i = \mathcal{M}(\phi_i) \phi_i$

The expectation is in fact the expectation for the case in which no measurement whatsoever (``identity operator'') is performed in the other subsystem.

Invert 1. We remark at this point that the ambiguity which arises when $\sum_i (\phi_i^0, \psi_i^0) \phi_i^0 = 0$ (see footnote, pg. — ) is unimportant for this representation, since any zero state can be regarded as the relative state, for the term $\psi_i^0 \phi_i^0$ will occur in (1.13) with sufficient zeros.

Invert 2.

There do not, in general, exist anything like a state for one subsystem of a composite system. The subsystems do not possess states independent of the states of the remainder of the system, so that the subsystem states are correlated. One can arbitrarily choose a state for one subsystem, and be led to the relative state for the other subsystem. Thus we are faced with a fundamental relativity of states, implied by the formalism of composite systems. It is meaningless to ask the absolute state of a subsystem -- one can only ask the state relative to a given state of the remainder of the system.
and the conditional distributions \( p_i^j \) and \( p_i^\lambda \) are:

\[
\begin{align*}
 p_i^j &= P(\varphi_i^j \text{ conditional on } \theta_i^j) = \frac{p_{i,j}}{p_i} \\
 p_i^\lambda &= P(\theta_i^\lambda \text{ conditional on } \varphi_i^\lambda) = \frac{p_{i,\lambda}}{p_i}
\end{align*}
\]

We next define Conditional Expectation of an operator \( A \) on \( S_1 \) conditioned on \( \theta_i^\lambda \) in \( S_2 \), \( \text{Exp}^{\theta_i^\lambda}[A] \):

\[
\text{Exp}^{\theta_i^\lambda}[A] = \sum_{i,j} \lambda_{ij} p_i^j = \frac{1}{p_i} \sum_{i,j} p_{i,j} \lambda_{ij}
\]

\[
= \frac{1}{p_i} \sum_{i,j} \lambda_{ij} |(\phi_i^j, \psi^j)|^2 = \sum_{i,j} |(\phi_i^j, \psi^j)|^2 (\varphi_i^j A \varphi_i^j)
\]

And finally we define the marginal expectation of \( A \) on \( S_2 \):

\[
\text{Exp}[A] = \sum_{i,j} p_i \lambda_{ij} = \sum_{i,j} \lambda_{ij} p_{i,j} = \sum_{i,j} |(\phi_i^j, \psi^j)|^2 (\varphi_i^j A \varphi_i^j)
\]

We shall now introduce projection operators to get more forms of the conditional and marginal quantities which will show more clearly the degree of dependence of these quantities upon the chosen basis. Let [\( \Phi_i \)] and [\( \Theta_i \)] be projections on \( \varphi_i^j \) in \( S_1 \) and \( \theta_i^\lambda \) in \( S_2 \), respectively, and let \( I_S^1, I_S^2 \) be the identity operators in \( S_1 \) and \( S_2 \).

\[
\text{Making use of the identity } \psi_S^i = \sum_{k,l,m} (\Phi_i^k, \phi_S^k) \Phi_i^k \text{ for any complete set of } \Phi_i^k
\]

\[
\langle \Theta_i \mid \theta_j^\lambda \rangle \psi_S^i = (\psi_S^i \mid \Theta_i \rangle \psi_S^i) = \left( \sum_{k,l,m} (\Phi_i^k, \phi_S^k) \Phi_i^k \right) \left( \sum_{k,l,m} (\Theta_i^l, \phi_S^l) \Theta_i^l \right)
\]

\[
= \sum_{k,l,m} (\phi_i^j, \psi^j) (\phi_i^k, \psi^k) \delta_{i}^{kl} \delta_{i}^{mn} = (\phi_i^j, \psi^j) (\phi_i^k, \psi^k) = p_i^j
\]
So that the joint distribution is given simply by
\[
\psi_{j} = \sum_{j} p_{j} = \sum_{j} \langle \phi_{j} \mid \psi \rangle \psi_{j}^{s} = \langle \mathcal{H} \mid \sum_{j} \langle \phi_{j} \mid \psi \rangle \rangle \psi_{j}^{s} = \langle \mathcal{H} \mid \mathcal{I} \rangle \psi_{j}^{s}
\]

and we see that the marginal distribution over the \( S_{j} \) is independent of the set \( \{ S_{j} \} \) chosen in \( S_{2} \). This has the consequence in the ordinary interpretation that the measurement expected outcome of measurement in one subsystem of a composite system is not influenced by the choice of quantity to be measured in the other subsystem, so long as the result of such a measurement remains unknown. Therefore, this gives the correct expectation in the case that no measurement (identity) is performed in the other subsystem. Thus no measurement in \( S_{2} \) can affect the expected outcome of a measurement in \( S_{1} \) so long as the result of the \( S_{2} \) measurement remains unknown. The case is quite different, however, if this result is known, and we must turn to the conditional distributions and expectations in such a case.

We now introduce the concept of relative state function, which will play a central role in our interpretation. Consider a composite system \( S = S_{1} + S_{2} \) in state \( \psi^{s} \). To every state \( \mathcal{N} \) of \( S_{2} \) we associate a state \( \psi_{\text{rel}}^{s} \) of \( S_{1} \) called the relative state in \( S_{1} \) for \( \mathcal{N} \), the
\[
(1.9) \text{Def: } \psi_{\text{rel}}^{s} = \mathcal{N} \sum_{j} \langle \phi_{j} \mid \psi^{s} \rangle \psi_{j}^{s}
\]
where \( \{ \phi_{j} \} \) is any complete orthonormal set in \( S_{2} \) and \( \mathcal{N} \) a normalization constant.
The first property of $\Psi_1$ is its uniqueness, that is, its dependence upon the choice of the basis $\{\phi_i\}$ is only apparent. To prove this, choose another basis $\{\phi_i^*\}$ with $\phi_i = \sum_k b_{ik} \phi_k$. Then $\sum_i b_{ik}^* b_{ik} = \delta_{jk}$, and:

$$\sum_a (\phi_a \psi^q)_{\phi_i} = \sum_a (\sum_j b_{aj}^* \phi_j \psi^q) (\sum_k b_{ik} \phi_k)$$

$$= \sum_j (\sum_k (b_{aj}^* b_{ik}) \phi_j \psi^q) \phi_k = \sum_k \sum_j (\phi_j \psi^q) \phi_k$$

The second property of the relative state $\Psi_1$, which justifies its name, is that $\Psi_1$ correctly gives the conditional expectation of all operators in $\mathfrak{g}_2$, conditioned by $\Theta_i$ in $\mathfrak{g}_2$. As before, let $A$ be an operator in $\mathfrak{g}_2$ with eigenstates $\phi_i$. Then:

$$\langle A \rangle_{\Psi_1} = (\psi_{\Theta_1}, A_{\Psi_1} \psi_{\Theta_1})$$

$$= (N \sum_j (\phi_j \psi^q)_{\phi_i} A N \sum_m (\phi_m \psi^q)_{\phi_i})$$

$$= N^2 \sum_{j,m} (\phi_j \psi^q)^* (\phi_m \psi^q) \lambda_j \delta_{j,m}$$

At this point, $N^2$ can be conveniently evaluated by using (1.10) to compute $\langle I \rangle_{\Psi_1} = N^2 \sum_j \delta_j = N^2 \langle \phi_1 \rangle = 1$ so that $N^2$

$$N^2 = \frac{1}{\delta_j}$$
Substitution of (1.11) in the development of (1.10) yields:

$\langle A \rangle \psi^{\Theta_3} = \frac{1}{P_3} \sum \lambda_i P_{i\Theta_3} = \sum \lambda_i P_i^3 = \exp \Theta_3 [A]$ (1.12)

And we have proved that the conditional expectations of operators are given by the relative states. (This of course includes the conditional distributions themselves, since they may be obtained as expectations of projection operators.)

Another worthwhile property of the relative states $\psi_{rel}^{\Theta_3}$ is that it depends only upon the single state $\Theta_3$ in $S_3$, and is hence independent of any choice of basis in the orthogonal space to $\Theta_3$.

An important representation of a composite system state $\psi^5$ in terms of an orthonormal set $\{\Theta_j, \varphi_j\}$ and the set of relative states $\{\psi_{rel}^{\Theta_3}\}$ is:

$\psi^5 = \sum \left( \langle \varphi_j | \psi^5 \rangle \varphi_j \right) \Theta_j = \sum \left[ \sum \left( \langle \varphi_j | \psi^5 \rangle \varphi_j \right) \right] \Theta_j$

$= \sum \frac{1}{N_j} \left[ N_j \sum \langle \varphi_j | \psi^5 \rangle \varphi_j \right] \Theta_j$

$= \sum \frac{1}{N_j} \psi_{rel}^{\Theta_j} \Theta_j$ (1.13)

Thus, for any orthonormal set in one subsystem, the single state of the composite system is a superposition of elements consisting of a state of the given set and its relative state in the other subsystem. We notice further, that a particular element $\psi_{rel}^{\Theta_3 \Theta_2}$ is quite independent of the choice of basis $\{\Theta_j, \varphi_j\}$ for the orthogonal space of $\Theta_2$, since $\psi_{rel}^{\Theta_3}$ depends only on $\Theta_3$ and not on the other $\Theta_k$ for $k \neq 3$.}

\text{Footnote} 2

\text{Here we clean up any ambiguity of $\psi_{rel}^{\Theta_3}$, in the case any state can be $\psi_{rel}^{\Theta_3}$ but it is combined with $\Theta_3$ in some way. (1.13).}
Now that we have found states which correctly give conditional expectations, we might inquire as to whether there exist states which give marginal expectations. The answer is always negative. Let us compute the marginal expectation of $A$ in $\gamma_j$ using representation (1.13):

\[
\text{Exp} [A] = \langle A \gamma_j \rangle = \left( \sum_{i \neq j} \frac{1}{N_j N_k} y_{i j}^* \rho_i y_{k j} \right) A \gamma_j \sum_{k \neq j} \frac{1}{N_j N_k} y_{k j}^* \rho_k y_{j j} \gamma_j
\]

\[
= \sum_{i \neq j} \frac{1}{N_j N_k} \left( y_{i j}^* \rho_i A y_{k j} \right) \gamma_j
\]

\[
= \sum_{i \neq j} \frac{1}{N_j} \left( y_{i j}^* A y_{k j} \right) \gamma_j = \sum_{i \neq j} \langle A \rangle y_{i j}^* y_{k j} \gamma_j
\]

Now suppose that there exists a state in $\gamma_j$ which correctly gives the marginal expectation (1.14) for all operators $A$ (i.e., $\text{Exp} [A] = \langle A \gamma_j \rangle$). One such operator is $\left[ y_{i j}^* \right]$, the projection on $y_{i j}$, for which $\langle A \rangle y_{i j} = 1$. But, from (1.14), $\text{Exp} [y_{i j}^*] = \sum_{i \neq j} \langle A \rangle y_{i j}^* y_{k j} \gamma_j$, which is $< 1$ unless $p_i = 0$ or $y_{i j}^* y_{k j} = 0$ (for all $j$), a condition which is not generally true. Therefore there exists in general no state for $\gamma_j$ which correctly gives marginal expectations.

Even though there is no state describing marginal expectations, there is a mixture of states, namely the states $y_{i j}^* y_{k j}$ weighted with $p_i$, which yields the desired expectations. The distinction between a mixture, $\Pi_j$, of states $\gamma_j$, weighted by $p_i$, and a pure state $\gamma_j$ which is a superposition, $\gamma_j = \sum_{i \neq j} \langle A \rangle y_{i j}^* y_{k j}$, is that there are no interference phenomena between the various states of a mixture. The expectation of operator $A$ for the mixture is $\text{Exp}^\Pi [A] = \sum_{i \neq j} p_i \langle A \rangle y_{i j}^* y_{k j} = \sum_{i \neq j} \langle A \rangle y_{i j}^* y_{k j}$, while the expectation for the pure state $\gamma_j$. 

\[
\text{Exp}^\Pi [A] = \sum_{i \neq j} p_i \langle A \rangle y_{i j}^* y_{k j} = \sum_{i \neq j} \langle A \rangle y_{i j}^* y_{k j}
\]
\[ \langle \psi | \mathbf{A} | \psi \rangle = \sum_{i,j} \alpha_i^* \alpha_j \langle \psi_i | \mathbf{A} | \psi_j \rangle \]

which is not the same as the mixture with weights \( p_i = \alpha_i^* \alpha_i \)

due to the presence of the interference terms \( \langle \psi_i | \mathbf{A} | \psi_j \rangle \) for \( i \neq j \).

It is convenient to represent such a mixture by a density matrix, \( \rho \), if the mixture consists of

the states \( | \psi_i \rangle \) weighted by \( p_i \), and if we are working in a basis consisting of the complete orthonormal set \( \{ \phi_j \} \)

where \( \psi_i = \sum_j a_i^j \phi_j \), then we define the elements of the density matrix for the mixture to be:

\[
\rho_{kj} = \sum_j p_j a_k^j a_j^* \tag{1.15}
\]

If \( \mathbf{A} \) is any operator, with matrix rep. \( A_{kl} = \langle \phi_k | \mathbf{A} | \phi_l \rangle \) in the chosen basis, then its expectation for the mixture is:

\[
\langle \psi | \mathbf{A} | \psi \rangle = \sum_j p_j (\psi_j | \mathbf{A} | \psi_j) = \sum_j p_j \left[ \sum_{kl} a_k^j a_l^* \langle \phi_k | \mathbf{A} | \phi_l \rangle \right]
\]

\[
= \sum_j \left( \sum_k p_k a_k^j a_j^* \right) \langle \phi_k | \mathbf{A} | \phi_j \rangle = \sum_{j,k} p_{kj} A_{jk}
\]

\[
= \text{Trace}(\rho \mathbf{A})
\]

Therefore my mixture is adequately represented by a density matrix, \( \rho \). Note also that \( \rho^* = \rho \), so that \( \rho \) is Hermitian.

Let us now find the density matrices \( \rho_{1\cdot} \) and \( \rho_{2\cdot} \)

to describe the subsystems \( S_1 \) and \( S_2 \) of a system \( S = S_1 \otimes S_2 \) in state \( \psi \).

Further, let us choose the orthonormal basis \( \{ \phi_k \} \) and \( \{ \phi'_k \} \) in \( S_1 \) and \( S_2 \) respectively, and let \( \mathbf{A} \) be an operator in \( S_1 \), \( \mathbf{B} \) an operator in \( S_2 \). Then:
\[
\text{Exp}[A] = \left< A \right> = \sum_{x_i} (\mathbf{x}_i \cdot \mathbf{y}_i) \mathbf{x}_i \cdot \mathbf{y}_i = \text{Trace} \left( \rho^S \mathbf{A} \right)
\]

where we have defined \( \rho^S \), in the \( \{ \mathbf{y}_i \} \) basis to be:

\[
\rho^S_{\mathbf{e}_i} = \sum_{j} (\mathbf{e}_i \cdot \mathbf{y}_j) \mathbf{e}_i \cdot \mathbf{y}_j
\]

In similar fashion we find that \( \rho^S \) is in \( \{ \mathbf{w}_i \} \) basis:

\[
\rho^S_{\mathbf{e}_i} = \sum_{j} (\mathbf{e}_i \cdot \mathbf{w}_j) \mathbf{e}_i \cdot \mathbf{w}_j
\]

It can be easily shown that here again, the dependence of \( \rho^S \) upon the choice of basis \( \{ \mathbf{y}_i \} \) or \( \{ \mathbf{w}_i \} \) is only apparent.

In summary, we have found in this section that a state of a composite system leads to joint distribution functions, which are generally not independent, conditional distributions and expectations can be obtained from relative states, and marginal distributions and expectations are given by density matrices.

There is, in general, no single state of a subsystem, which is independent of the state of the other subsystem. Subsystems do not possess states independently, only relations between states of subsystems -- i.e., they are correlated. One can arbitrarily choose a state for one subsystem, and be led to relative states for the other. There is no true separability of states.
implied by the formalism of composite systems. It is meaningless to ask the state of a subsystem; one can only ask the state relative to a given state of the remainder.